Model of attrition process in the presence of decoys

M. CHUDY
mchudy@wat.edu.pl

Military University of Technology, Faculty of Cybernetics
Kaliskiego Str. 2, 00-908 Warsaw, Poland

Mutual destroying process on a battlefield has classically been modeled without accounting for the possible presence of false targets. Following [6] we include into consideration the presence of decoys. Attrition of human decoys is to be strenuously avoided for humanitarian reasons, but also because of its broad impact on world opinion. False targets should be taken into consideration in the formulation of the target assignment problems. We formulate bicriteria assignment problem. First of them maximizes expected value of destroyed correctly detected targets and the second minimizes expected value of destroyed incorrectly detected targets. The resulting problem provides the set of compromise solutions.
Each of the considered assignment problems belong to the class of general assignment problem which does not contain totally unimodular matrix factors.

Keywords: mathematical modeling, attrition process, assignment problem, decoy.

1. Introduction

Following [4] we will keep the same main notations adding only necessary ones.
Our intention is to give such a description of an attrition process that could be useful in the construction of computer combat simulator software.
Let us denote by
$\mathcal{N}^A(t)$ – the set of numbers of objects that belong to side A at time t (usually it is unknown for opponent),
$\mathcal{N}^B(t)$ – the set of numbers of objects that belong to side B at time t (usually it is unknown for opponent),
$S(t) = (S^A(t), S^B(t))$ – the state of both sides at time t.
We consider two-sided battle and assume that both sides use only two-state $(1,0)$ objects, where:
1 – denotes the state when an object is undestroyed
0 – denotes the state when an object is destroyed.
It means that

$$S^A(t) = \begin{pmatrix} (S_1^A(t), S_2^A(t), \ldots, S_{\mathcal{N}^A(t)}^A(t)), \\ S_i^A(t) \in \{0,1\},\ i \in \mathcal{N}^A(t) \end{pmatrix}$$

$$S^B(t) = \begin{pmatrix} (S_1^B(t), S_2^B(t), \ldots, S_{\mathcal{N}^B(t)}^B(t)), \\ S_i^B(t) \in \{0,1\},\ i \in \mathcal{N}^B(t) \end{pmatrix}$$

We denote by

$$N^A(t) = \{i \in \mathcal{N}^A(t) : S_i^A(t) = 1\}$$

$$N^B(t) = \{j \in \mathcal{N}^B(t) : S_j^B(t) = 1\}$$

the sets of undestroyed objects of side A and B respectively.
In general, the both sides cannot recognize all of the objects belonging to the opponent.
Therefore we should define the following sets:
$N_B^A(t)$ – the set of numbers of undestroyed objects of side A which are detected by side B at time t,
$N_A^B(t)$ – the set of numbers of undestroyed objects of side B which are detected by side A at time t.
The object which index belongs to the set $N_B^A(t)$ or $N_A^B(t)$ can be taken into account in the control of an attrition process.
We assume that there are incorrectly detected objects inside the sets $N_B^A(t)$ and $N_A^B(t)$.
Our mistakes and opponent mistakes in detecting targets can by described by introducing the following independent random variables:

$$R_j^B = \begin{cases} 1 & \text{when object } j\text{-th of side } B \\ & \text{is correctly detected as} \\ & \text{a target by side } A \\ 0 & \text{in other cases} \end{cases}$$

$$R_i^A = \begin{cases} 1 & \text{when object i-th of side } A \text{ is correctly detected as a target by side } B \\ 0 & \text{in other cases} \end{cases}$$

We also assume that

$$R_j^B = \begin{cases} 1 & \text{with probability } P_j^B \\ 0 & \text{with probability } 1 - P_j^B. \end{cases}$$

and

$$R_i^A = \begin{cases} 1 & \text{with probability } P_i^A \\ 0 & \text{with probability } 1 - P_i^A. \end{cases}$$

One can observe that both sides of the battle should respect the following rules:

– maximize the losses of correctly detected targets,
– minimize the losses of incorrectly detected targets.

2. Assignment problem of targets in the presence of decoys

It is generally known that an object can destroy other object belonging to the opponent side if and only if it possess enough amount of destroying resources.

Let us denote by

$Z_{ij}^A(t)$ – total amount of resources belonging to the i-th object of side A that can be used to destroy the j-th object of side B at time,

hereof we can define

$$\overline{N}_j^A(t) = \left\{ i \in \mathcal{N}^A(t) : Z_{ij}^A(t) \geq \overline{Z}_{ij}^A(t) > 0 \right\}$$

which means the set of indices of the objects belonging to side A that can destroy the j-th objects of side B at time t

where:

$\overline{Z}_{ij}^A(t)$ – the threshold value of resources at time t,

and

$$\underline{N}_i^B(t) = \left\{ j \in N_A^B(t) : Z_{ij}^A(t) \geq \overline{Z}_{ij}^A(t) > 0 \right\}$$

which means the set of indices of the objects belonging to side B that can be destroyed by i-th object of side A at time t.

Moreover we denote by

$b_{ij}^B(t)$ – the loss of potential of side B when the j-th object is destroyed by the i-th object of side A at time t under condition that j-th object was correctly detected as a target,

D_{ij}^B – the loss of side B when its j-th objects is destroyed by the i-th object of side A under condition that j-th object was incorrectly detected as a target,

$\tilde{N}_A^B(t)$ – the set of indices of objects belonging to the side B, which got priority to be destroyed by side A at time t,

and decision variables

$$x_{ij}(t) = \begin{cases} 1 & \text{when object i-th of side A is assigned to destroy object j-th of side B} \\ 0 & \text{in other cases} \end{cases}$$

We assume that the target assignment problems will be solved by two sides at these moments when the state $S(t)$ changes or at other times determined by decision makers.

Let $(t_k)_{k=0,1,2,\ldots}$ denotes the sequence of the moments when at least one of the side changes its assignment of fire (targets).

At the moment t_k the set $\Omega_A(t_k)$ of feasible solution of side A is determined by following constrains:

$$\sum_{i \in \overline{N}_j^A(t_k)} x_{ij}(t_k) = 1, \quad j \in \tilde{N}_A^B(t_k) \subset N_A^B(t_k) \tag{1}$$

$$\sum_{i \in \overline{N}_j^A(t_k)} x_{ij}(t_k) \leq 1, \quad j \in N_A^B(t_k) \setminus \tilde{N}_A^B(t_k) \tag{2}$$

$$\sum_{j \in \underline{N}_i^B(t_k)} x_{ij}(t_k) \leq 1, \quad i \in \mathcal{N}^A(t_k) \tag{3}$$

$$x_{ij}(t_k) \in \{0, 1\} \tag{4}$$

hence

$$\Omega_A(t_k) = \left\{ x(t_k) = (x_{ij}(t_k)) \text{ that satisfies } (1),(2),(3),(4) \right\}$$

(5)

To take into account the presence of decoys in the battle we must consider the following objective functions:

- expected value of the military potential loss of side B after the destruction of correctly detected targets,
- expected value of loss of side B after the destruction of incorrectly detected targets.

First of them can be described as follows

$$F_1^A\left(x\left(t_k\right)\right)= = E\left(\sum_{j\in N_A^B(t_k)}\ \sum_{i\in \bar{N}_j^A(t_k)} R_j^B b_{ij}^B\left(t_k\right) x_{ij}\left(t_k\right)\right) = = \sum_{j\in N_A^B(t_k)}\ \sum_{i\in \bar{N}_j^A(t_k)} P_j^B b_{ij}^B\left(t_k\right) x_{ij}\left(t_k\right) \qquad (6)$$

the second function has the form

$$F_2^A\left(x\left(t_k\right)\right)= = E\left(\sum_{j\in N_A^B(t_k)}\ \sum_{i\in \bar{N}_j^A(t_k)} \left(1-R_j^B\right) D_{ij}^B x_{ij}\left(t_k\right)\right) = = \sum_{j\in N_A^B(t_k)}\ \sum_{i\in \bar{N}_j^A(t_k)} \left(1-P_j^B\right) D_{ij}^B x_{ij}\left(t_k\right) \qquad (7)$$

According to the accepted rules we are obligated to solve bicriteria assignment problem that consist of two subproblems:

$$F_1^A\left(x^1\left(t_k\right)\right)= \max_{x(t_k)\in\Omega_A(t_k)} F_1^A\left(x\left(t_k\right)\right) \qquad (8)$$

$$F_2^A\left(x^2\left(t_k\right)\right)= \min_{x(t_k)\in\Omega_A(t_k)} F_2^A\left(x\left(t_k\right)\right) \qquad (9)$$

Many universal method suitable in this case are presented for example in [2].

We propose to find a compromise solution using normalized ideal point (1,1) for objective functions.

The resulting problem for finding compromise solution has the form

$$\min_{x(t_k)\in\Omega_A(t_k)}\left(\frac{F_1^A\left(x^1\left(t_k\right)\right)-F_1^A\left(x\left(t_k\right)\right)}{F_1^A\left(x^1\left(t_k\right)\right)}+ + \frac{F_2^A\left(x^2\left(t_k\right)\right)-F_2^A\left(x\left(t_k\right)\right)}{F_2^A\left(x^2\left(t_k\right)\right)}\right) \qquad (10)$$

The problems (8), (9) and (10) are NP-hard problems belonging to the class of general assignment problems, which can be solved in approximate way applying known methods [5].

We notice that at the moment t_k we have to solve three problems: (8), (9) and resulting problems (10).

By analogy we can describe and formulate suitable problems for side B.

Introducing similar denotation:

$Z_{ji}^B(t)$, $\bar{N}_i^B(t)$, $\bar{Z}_{ji}^B(t)$, $\underline{N}_j^A(t)$, $b_{ji}^A(t)$, $D_{ji}^A(t)$, $\tilde{N}_B^A(t)$

decision variables

$$y_{ij}(t)=\begin{cases}1 \text{ when object } j\text{-th of side } B \text{ is assigned to destroy object } i\text{-th of side } A\\ 0 \text{ in other cases}\end{cases}$$

the set $\Omega_B\left(t_k\right)$ of feasible solutions of side B is defined by the following constrains at time t_k :

$$\sum_{j\in \bar{N}_i^B(t_k)} y_{ji}\left(t_k\right)=1\ ,\ i\in \tilde{N}_B^A\left(t_k\right)\subset N_B^A\left(t_k\right) \qquad (11)$$

$$\sum_{j\in \bar{N}_i^B(t_k)} y_{ji}\left(t_k\right)\le 1\ ,\ i\in N_B^A\left(t_k\right)\setminus \tilde{N}_B^A\left(t_k\right) \qquad (12)$$

$$\sum_{i\in \underline{N}_j^A(t_k)} y_{ji}\left(t_k\right)\le 1\ ,\ j\in \mathcal{N}^B\left(t_k\right) \qquad (13)$$

$$y_{ji}\left(t_k\right)\in\{0,1\} \qquad (14)$$

hence

$$\Omega_B\left(t_k\right)=\left\{\begin{matrix} y=\left(y_{ji}\left(t_k\right)\right) \text{ that satisfies}\\ (11), (12), (13), (14)\end{matrix}\right\}$$

The elements of bicriteria function are:

$$F_1^B\left(y\left(t_k\right)\right)= = E\left(\sum_{i\in N_B^A(t_k)}\ \sum_{j\in \bar{N}_i^B(t_k)} R_i^A b_{ji}^A\left(t_k\right) y_{ji}\left(t_k\right)\right) = = \sum_{i\in N_B^A(t_k)}\ \sum_{j\in \bar{N}_i^B(t_k)} P_i^A b_{ji}^A\left(t_k\right) y_{ji}\left(t_k\right), \qquad (15)$$

$$F_2^B\left(y\left(t_k\right)\right)= = E\left(\sum_{i\in N_B^A(t_k)}\ \sum_{j\in \bar{N}_i^B(t_k)} \left(1-R_i^A\right) D_{ji}^A\, y_{ji}\left(t_k\right)\right) = = \sum_{i\in N_B^A(t_k)}\ \sum_{j\in \bar{N}_i^B(t_k)} \left(1-P_i^A\right) D_{ji}^A\, y_{ji}\left(t_k\right), \qquad (16)$$

The subproblems have the forms

$$F_1^B\left(y^1(t_k)\right)=\max_{y(t_k)\in\Omega_B(t_k)} F_1^B\left(y(t_k)\right) \qquad (17)$$

$$F_2^B\left(y^2(t_k)\right)=\min_{y(t_k)\in\Omega_B(t_k)} F_2^B\left(y(t_k)\right). \qquad (18)$$

The resulting problem for finding compromise solution with normalized ideal point (1, 1) for objective functions can be formulated in such a way:

$$\min_{y(t_k)\in\Omega_B(t_k)}\left(\frac{F_1^B\left(y^1(t_k)\right)-F_1^B\left(y(t_k)\right)}{F_1^B\left(y^1(t_k)\right)}+\frac{F_2^B\left(y^2(t_k)\right)-F_2^B\left(y(t_k)\right)}{F_2^B\left(y^2(t_k)\right)}\right) \qquad (19)$$

The assignment problems formulated above belong to class of NP – hard problems. So, even for small size of problem to obtain optimal solution is difficult. In practice approximate solutions are accepted. Known examples of methods that solved assignment problems in approximate way are presented in [5].

In all considered assignment problems we take into account the following circumstances:

potential of each of the sides is additive

at most one object belonging to one of the side can destroy one object belonging to the opposite side the losses of both sides are additive.

The objective functions reflect the advantaqges that both sides are going to achieve when the assignment $x(t_k)$ and $y(t_k)$ are realized.

The method of calculation of t_k is precisely described in [4] and can be applied to the considered problem.

When we prefer the first criteria to the second, both bicriteria problems (8), (9) and (17), (18) can be solved in hierarchical way.

The procedure of the hierarchical method for side A can be presented as follows.

1. Compute set

$$\Omega_A^*(t_k)=\left\{x^1(t_k):\max_{x(t_k)\in\Omega_A(t_k)} F_1^A(t_k)\right\}.$$

2. Compute solution $x^*(t_k)$ that satisfies

$$F_2^A(t_k)=\min_{x(t_k)\in\Omega_A^*(t_k)} F_2^A(t_k).$$

Vector $x^*(t_k)$ is the hierarchical solution for side A.

By analogy we can compute hierarchical solution $y^*(t_k)$ for side B.

To obtain some heuristic solution of bicriteria problems (8), (9) or (17), (18) one can apply approach proposed in [8].

3. Conclusions

The presented approach to the problem of presence of decoys in military activity is suitable only to selected form of combat process. The problem is to possess information at time t_k about the value of considered parameters. The system of detecting targets should be significantly extended in these cases but it increases the cost of the operation.

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Model procesu ubywania w sytuacji istnienia obiektów pozornych

M. CHUDY

W klasycznych modelach walki zwykle nie uwzględnia się obiektów źle zidentyfikowanych lub pozornych mających na celu zmylenie przeciwnika. W ślad za propozycją [6] taka obecność zostanie włączona do sformułowań zadań przydziału. Sformułowane zostanie dwukryterialne zadanie przydziału, w którym maksymalizuje się oczekiwaną wartość poprawnie zidentyfikowanych obiektów przeciwnika oraz minimalizuje oczekiwaną wartość niepoprawnie zidentyfikowanych obiektów. Takie sytuacje dotyczą działań bojowych w operacjach nietypowych, gdzie niszczenie obiektów, które nie są obiektami militarnymi, jest negatywnie oceniane z humanitarnego punktu widzenia.
W problemie wynikowym proponuje się wyznaczenie rozwiązania kompromisowego przy pomocy znanych metod.
Każdy ze sformułowanych problemów przydziału należy do klasy uogólnionych zadań przydziału i może nie posiadać unimodularnej macierzy współczynników.

Słowa kluczowe: modelowanie matematyczne, proces ubywania, zadania przydziału, obiekty pozorne.