Model of attrition process control

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The essence of the presented attrition process control relies on the solving defined sequence of target assignment problems at specific moments. The sequence of these moments is obtained for both sides of the battle. The model takes into account the changes of the number of means and targets as well as changes in environmental conditions. It is reflected in the parameters of problems. Each of the considered assignment problems belongs to the class of general assignment problems which does not contain totally unimodular matrix factors.

Keywords: mathematical modeling, attrition process, assignment problem.

1. Introduction and main notation

The control of the attrition process is the main part of the combat model. The main step in this process is to solve many specific assignment problems. Such an approach is also presented for example in [1], [3], [7].

Our objective is to describe the attrition process control that could be useful in the construction of computer combat simulator software.

Let us denote by

$\mathcal{N}^{\mathbf{A}}(t)$ – the set of numbers of objects that belong to side $\mathbf{A}$ at time t (usually it is unknown to the opponent),

$\mathcal{N}^{\mathbf{B}}(t)$ – the set of numbers of objects that belong to side $\mathbf{B}$ at time t (usually it is unknown to the opponent),

$s(t) = \left(s^{\mathbf{A}}(t), s^{\mathbf{B}}(t)\right)$ – the state of both sides at time t.

We consider two-sided battle and assume that both sides use only two-state $(1,0)$ objects, where

1 – denotes the state when an object is undestroyed.

0 – denotes the state when an object is destroyed.

It means that

$$s^{\mathbf{A}}(t)=\left(s_1^{\mathbf{A}}(t), s_2^{\mathbf{A}}(t), \ldots, s^{\mathbf{A}}_{\mathcal{N}^{\mathbf{A}}(t)}(t), s_i^{\mathbf{A}}(t) \in \{0,1\}, i \in \mathcal{N}^{\mathbf{A}}(t)\right) \tag{1}$$

$$s^{\mathbf{B}}(t)=\left(s_1^{\mathbf{B}}(t), s_2^{\mathbf{B}}(t), \ldots, s^{\mathbf{B}}_{\mathcal{N}^{\mathbf{B}}(t)}(t), s_i^{\mathbf{B}}(t) \in \{0,1\}, i \in \mathcal{N}^{\mathbf{B}}(t)\right) \tag{2}$$

We denote by

$$N^{\mathbf{A}}(t) = \left\{ i \in \mathcal{N}^{\mathbf{A}}(t) : s_i^{\mathbf{A}}(t) = 1 \right\} \tag{3}$$

$$N^{\mathbf{B}}(t) = \left\{ j \in \mathcal{N}^{\mathbf{B}}(t) : s_j^{\mathbf{B}}(t) = 1 \right\} \tag{4}$$

the sets of undestroyed objects of side $\mathbf{A}$ and $\mathbf{B}$ respectively.

In general, both sides cannot recognize all of the objects belonging to the opponent.

Therefore, we should define the following sets:

$N_{\mathbf{B}}^{\mathbf{A}}(t)$ – the set of numbers of undestroyed objects of side $\mathbf{A}$ which are detected by side $\mathbf{B}$ at time t,

$N_{\mathbf{A}}^{\mathbf{B}}(t)$ – the set of numbers of undestroyed objects of side $\mathbf{B}$ which are detected by side $\mathbf{A}$ at time t.

The object whose index belongs to the set $N_{\mathbf{B}}^{\mathbf{A}}(t)$ or $N_{\mathbf{A}}^{\mathbf{B}}(t)$ can be taken into account in the control of an attrition process.

2. Targets assignment problem

An object can destroy another object belonging to the opponent side if and only if it possess enough destroying resources.

We denote by

$z_{ij}^{\mathbf{A}}(t)$ – total amount of resources belonging to the i-th object of side $\mathbf{A}$ that can be used to destroy the j-th object of side $\mathbf{B}$ at time t,

$z_{ij}^{\mathbf{B}}(t)$ – total amount of resources belonging to the j-th object of side $\mathbf{B}$ that can be used to destroy the i-th object of side $\mathbf{A}$ at time t.

We assume that the target assignment problems will be solved by two sides at the moments when the state $s(t)$ changes or at other times determined by decision makers.

Let $(t_k)_{k=0,1...}$ denote the sequence of the moments when at least one of the sides changes its assignment of fire (targets).

At each moment t_k the following assignment problems are being solved by **side** A

$$\max \sum_{j \in N_A^B(t_k)} \sum_{i \in \overline{N}_j^A(t_k)} b_{ij}^B(t_k) x_{ij}(t_k) \tag{5}$$

subject to

$$\sum_{i \in \overline{N}_j^A(t_k)} x_{ij}(t_k) = 1,\ j \in \tilde{N}_A^B(t_k) \subset N_A^B(t_k) \tag{6}$$

$$\sum_{i \in \overline{N}_j^A(t_k)} x_{ij}(t_k) \le 1,\ j \in N_A^B(t_k) \setminus \tilde{N}_A^B(t_k) \tag{7}$$

$$\sum_{j \in \underline{N}_i^B(t_k)} x_{ij}(t_k) \le 1,\ j \in \mathcal{N}^A(t_k) \tag{8}$$

$$x_{ij}(t_k) \in \{0,1\} \text{ for all i,j} \tag{9}$$

where:

$b_{ij}^B(t_k)$ – the loss of potential of side B when the j-th object of side B is destroyed by the i-th object of side A at time t_k,

$\tilde{N}_A^B(t_k)$ – the set of indices of the objects belonging to side B which are given priority to be destroyed by side A at time t_k,

$\overline{N}_j^A(t_k)$ – the set of indices of the objects belonging to side A that **can destroy** the j-th object of side B at time t_k.

It means that

$$\overline{N}_j^A(t_k) = \left\{ i \in \mathcal{N}^A(t_k) : Z_{ij}^A(t_k) \ge \overline{Z}_{ij}^A(t_k) > 0 \right\}$$

where:

$\overline{Z}_{ij}^A(t_k)$ – the threshold value of resources at time t_k,

$\underline{N}_i^B(t_k)$ – the set of indices of the objects belonging to side B that **can be destroyed** by the i-th object of side A at time t_k,

it means that

$$\underline{N}_i^B(t_k) = \left\{ j \in N_A^B(t_k) : Z_{ij}^A(t_k) \ge \overline{Z}_{ij}^A(t_k) > 0 \right\}$$

and by **side** B

$$\max \sum_{i \in N_B^A(t_k)} \sum_{j \in \overline{N}_i^B(t_k)} b_{ij}^A(t_k) y_{ji}(t_k) \tag{10}$$

subject to

$$\sum_{j \in \overline{N}_i^B(t_k)} y_{ji}(t_k) = 1,\ i \in \tilde{N}_B^A(t_k) \subset N_B^A(t_k) \tag{11}$$

$$\sum_{j \in \overline{N}_i^B(t_k)} y_{ji}(t_k) \le 1,\ i \in N_B^A(t_k) \setminus \tilde{N}_B^A(t_k) \tag{12}$$

$$\sum_{i \in \underline{N}_j^A(t_k)} y_{ji}(t_k) \le 1,\ j \in \mathcal{N}^B(t_k) \tag{13}$$

$$y_{ji}(t_k) \in \{0,1\} \text{ for all i, j} \tag{14}$$

where:

$b_{ij}^A(t_k)$ – the loss of potential of side A when the i-th object of side A is destroyed by the j-th object of side B at time t_k,

$\tilde{N}_B^A(t_k)$ – the set of indices of the objects belonging to side A which are given priority to be destroyed by side B at time t_k,

$\overline{N}_i^B(t_k)$ – the set of indices of the objects belonging to side B that **can destroy** the i-th object of side A at time t_k,

it means that

$$\overline{N}_i^B(t_k) = \left\{ j \in \mathcal{N}^B(t_k) : Z_{ji}^B(t_k) \ge \overline{Z}_{ji}^B(t_k) > 0 \right\}$$

where:

$\overline{Z}_{ji}^B(t_k)$ – the threshold value of resources at time t_k,

$\underline{N}_j^A(t_k)$ – the set of indices of the objects belonging to side A that **can be destroyed** by the j-th object of side B at time t_k,

it means that

$$\underline{N}_j^A(t_k) = \left\{ i \in N_B^A(t_k) : Z_{ji}^B(t_k) \ge \overline{Z}_{ji}^B(t_k) > 0 \right\}$$

In both assignment problems we take into account the following circumstances:

- potential of each of the sides is additive;
- the commanders of each of the sides give the priority at time t_k to destroy opponent objects;
- there is no obligation to destroy all of the detected objects belonging to the opponent side, at time t_k;
- at most one object belonging to one of the sides can destroy one object belonging to the opposite side.

The last circumstance significantly reduces computations.

The objective functions reflected the advantages that both sides are going to achieve when assignments x and y are realized respectively.

The assignment problems formulated above belong to the class NP – hard problems. So, it is difficult to achieve an optimal solution even for small-sized problem.

In practice approximate solutions are accepted. Known examples of methods that solved assignment problems are presented in [4], [6], [8].

3. Calculation of t_k

The character of assignment problem and its parameters can be used to estimate the quality of conducting the process.

We especially take into account facts such as:

- character of objective function;
- dependences among the following sets $N^A(t_k)$, $N^B_A(t_k)$, $\tilde{N}^A_i(t_k)$, $\overline{N}^A_i(t_k)$ and $N^B(t_k)$, $N^A_B(t_k)$, $\tilde{N}^A_B(t_k)$, $\overline{N}^B_j(t_k)$ respectively;
- accuracy of solving the assignment problem;
- total amount of resources that, each of the sides possesses at time t_k;
- character of destroying streams which are generated by the objects at time t_k.

The destroying streams can be described in the following way:

$$h^A_{ij}(t_{k-1},t)=\min\left\{Z^A_{ij}(t_{k-1}),M^A_{ij}(t_{k-1},t)\right\} \tag{15}$$

$$h^B_{ji}(t_{k-1},t)=\min\left\{Z^B_{ji}(t_{k-1}),M^B_{ji}(t_{k-1},t)\right\} \tag{16}$$

where

$M^A_{ij}(t_{k-1},t)$ – the number of shots that could be fired by i-th object of side A at j-th object of side B during the period $t-t_{k-1}$ with unlimited amounts of resources,

$M^B_{ji}(t_{k-1},t)$ – the number of shots that could be fired by j-th object of side B at i-th object of side A during the period $t-t_{k-1}$ with unlimited amount of resources.

Examples of function $M^A_{ij}(t_{k-1},t)$ are:

$$M^A_{ij}(t_{k-1},t)=\left[\frac{t-t_{k-1}}{t^A_{ij}}\right] \tag{17}$$

where:

t^A_{ij} – the interval between following shots fired by i-th object of side A at j-th object of side B, or

$$M^A_{ij}(t_{k-1},t)=M^A_{ij}(\tau) \tag{18}$$

where:

$M^A_{ij}(\tau)$ – the number of renewals at the period $\tau=t-t_{k-1}$ in general renewal process.

Analogously for side B.

In order to calculate the amount of resources at time t we introduce the following notations:

$\tau^A_{ij}(k)$ – moment of delivering k-th supply for i-th object of side A to destroy j-th object of side B,

$d^A_{ij}(k)$ – the amount of supply at time $\tau^A_{ij}(k)$ for i-th object of side A,

$\tau^B_{ji}(k)$ – moment of delivering k-th supply for j-th object of side B to destroy i-th object of side A,

$d^B_{ji}(k)$ – the amount of supply at time $\tau^B_{ji}(k)$ for j-th object of side B,

and

$$\overline{K}^A_{ij}(t)=\sup\left\{k:\tau^A_{ij}(k)<t\right\}$$

$$\underline{K}^A_{ij}(t)=\inf\left\{k:\tau^A_{ij}(k)\geq t\right\}$$

$$\overline{K}^B_{ji}(t)=\sup\left\{k:\tau^B_{ji}(k)<t\right\}$$

$$\underline{K}^B_{ji}(t)=\inf\left\{k:\tau^B_{ji}(k)\geq t\right\}$$

At time t from the period $t_{k-1}<t\leq t_k$ the amount of resources equals

$$Z^A_{ij}(t)=Z^A_{ij}(t_{k-1}) \text{ when } x_{ij}(t_{k-1})=0 \tag{19}$$

while $x_{ij}(t_{k-1})=1$ we obtain

$$Z^A_{ij}(t)=Z^A_{ij}(t_{k-1})-k^A_{ij}(t_{k-1},t) \tag{20}$$

for t that satisfies

$\left\{t_{k-1} < t \le t_k \text{ and } \tau_{ij}^{A}\left(\underline{K}_{ij}^{A}(t_{k-1})\right) \ge t_k\right\}$ or

$\left\{t_{k-1} < t < \min\left\{\tau_{ij}^{A}\left(\underline{K}_{ij}^{A}(t_{k-1})\right), t_k\right\} \text{ and } \tau_{ij}^{A}\left(\underline{K}_{ij}^{A}(t_{k-1})\right) < t_k\right\}$,

$$Z_{ij}^{A}(t) = Z_{ij}^{A}\left(\tau_{ij}^{A}\left(\underline{K}_{ij}^{A}(t_{k-1}) + r\right)\right) + d_{ij}^{A}\left(\underline{K}_{ij}^{A}(t_{k-1}) + r\right) + \\ - \min\left\{\begin{array}{c} Z_{ij}^{A}\left(\tau_{ij}^{A}\left(\underline{K}_{ij}^{A}(t_{k-1}) + r\right)\right) + d_{ij}^{A}\left(\underline{K}_{ij}^{A}(t_{k-1}) + r\right), \\ M_{ij}^{A}\left(\tau_{ij}^{A}\left(\underline{K}_{ij}^{A}(t_{k-1}) + r\right), t\right) \end{array}\right\} \quad (21)$$

for t that satisfies

$$\tau_{ij}^{A}\left(\underline{K}_{ij}^{A}(t_{k-1}) + r\right) \le t \le \min\left\{t_k, \tau_{ij}^{A}\left(\underline{K}_{ij}^{A}(t_k) + r + 1\right)\right\}$$

$r = \overline{0, \overline{K}_{ij}^{A}(t_k) - \underline{K}_{ij}^{A}(t_{k-1})}$ and $\tau_{ij}^{A}\left(\underline{K}_{ij}^{A}(t_{k-1})\right) < t_k$

where

$$Z_{ij}^{A}\left(\tau_{ij}^{A}\left(\underline{K}_{ij}^{A}(t_{k-1}) + r\right)\right) = Z_{ij}^{A}\left(\tau_{ij}^{A}\left(\underline{K}_{ij}^{A}(t_{k-1}) + r - 1\right)\right) + \\ + d_{ij}^{A}\left(\underline{K}_{ij}^{A}(t_{k-1}) + r - 1\right) + \\ - \min\left\{\begin{array}{c} Z_{ij}^{A}\left(\tau_{ij}^{A}\left(\underline{K}_{ij}^{A}(t_{k-1}) + r - 1\right)\right) + d_{ij}^{A}\left(\underline{K}_{ij}^{A}(t_{k-1}) + r - 1\right), \\ M_{ij}^{A}\left(\tau_{ij}^{A}\left(\underline{K}_{ij}^{A}(t_{k-1}) + r - 1\right), t\right) \end{array}\right\} \quad (22)$$

Analogously for side B

$$Z_{ji}^{B}(t) = Z_{ji}^{B}(t_{k-1}) \text{ when } y_{ji}(t_{k-1}) = 0, \quad t_{k-1} < t \le t_k \quad (23)$$

while $y_{ji}(t_k) = 1$ we obtain

$$Z_{ji}^{B}(t) = Z_{ji}^{B}(t_{k-1}) - h_{ji}^{B}(t_{k-1}) \quad (24)$$

for t that satisfies

$\left\{t_{k-1} < t \le t_k \text{ and } \tau_{ji}^{B}\left(\underline{K}_{ji}^{B}(t_{k-1})\right) \ge t_k\right\}$ or

$\left\{t_{k-1} < t < \min\left\{\tau_{ji}^{B}\left(\underline{K}_{ji}^{B}(t_{k-1})\right), t_k\right\} \text{ and } \tau_{ji}^{B}\left(\underline{K}_{ji}^{B}(t_{k-1})\right) < t_k\right\}$

$$Z_{ji}^{B}(t) = Z_{ji}^{B}\left(\tau_{ji}^{B}\left(\underline{K}_{ji}^{B}(t_{k-1}) + r\right)\right) + d_{ji}^{B}\left(\underline{K}_{ji}^{B}(t_{k-1}) + r\right) + \\ - \min\left\{\begin{array}{c} Z_{ji}^{B}\left(\tau_{ji}^{B}\left(\underline{K}_{ji}^{B}(t_{k-1}) + r\right)\right) + d_{ji}^{B}\left(\underline{K}_{ji}^{B}(t_{k-1}) + r\right), \\ M_{ji}^{B}\left(\tau_{ji}^{B}\left(\underline{K}_{ji}^{B}(t_{k-1}) + r\right), t\right) \end{array}\right\} \quad (25)$$

for t that satisfies

$$\tau_{ji}^{B}\left(\underline{K}_{ji}^{B}(t_{k-1}) + r\right) \le t < \min\left\{t_k, \tau_{ji}^{B}\left(\underline{K}_{ji}^{B}(t_k) + r - 1\right)\right\}$$

$r = \overline{0, \overline{K}_{ji}^{B}(t_k) - \underline{K}_{ji}(t_{k-1})}$ and $\tau_{ji}^{B}\left(\underline{K}_{ji}^{B}(t_{k-1})\right) < t_k$

where:

$$Z_{ji}^{B}\left(\tau_{ji}^{B}\left(\underline{K}_{ji}^{B}(t_{k-1}) + r\right)\right) = Z_{ji}^{B}\left(\tau_{ji}^{B}\left(\underline{K}_{ji}^{B}(t_{k-1}) + r - 1\right)\right) + \\ + d_{ji}^{B}\left(\underline{K}_{ji}^{B}(t_{k-1}) + r - 1\right) + \\ - \min\left\{\begin{array}{c} Z_{ji}^{B}\left(\tau_{ji}^{B}\left(\underline{K}_{ji}^{B}(t_{k-1}) + r - 1\right)\right) + d_{ji}^{B}\left(\underline{K}_{ji}^{B}(t_{k-1}) + r - 1\right), \\ M_{ji}^{B}\left(\tau_{ji}^{B}\left(\underline{K}_{ji}^{B}(t_{k-1}) + r - 1\right), t\right) \end{array}\right\} \quad (26)$$

We should also introduce

$$t_i^{A}(k) = \min\left\{t : \sum_{n=1}^{h_{ji}^{B}(t_{k-1}, t)} r_{ji}^{B}(n, t_{k-1}) = 1, y_{ji}(t_{k-1}) = 1\right\} \quad (27)$$

for $i \in N_B^{A}(t_{k-1})$

$$t_j^{B}(k) = \min\left\{t : \sum_{n=1}^{h_{ij}^{A}(t_{k-1}, t)} r_{ij}^{A}(n, t_{k-1}) = 1, x_{ij}(t_{k-1}) = 1\right\} \quad (28)$$

for $j \in N_A^{B}(t_{k-1})$

where:

$$r_{ji}^{B}(n, t_{k-1}) = \begin{cases} 1 & \text{with prob. } p_{ji}^{B}(n, t_{k-1}) \\ 0 & \text{with prob. } 1 - p_{ji}^{B}(n, t_{k-1}) \end{cases} \quad (29)$$

random variable which equals 1 when the j-th object of side B is destroyed by n-th unique shot the i-th object of side A since the time t_{k-1}, and 0 otherwise.

$$r_{ij}^{A}(n, t_{k-1}) = \begin{cases} 1 & \text{with prob. } p_{ij}^{A}(n, t_{k-1}) \\ 0 & \text{with prob. } 1 - p_{ij}^{A}(n, t_{k-1}) \end{cases} \quad (30)$$

random variable which equals 1 when the i-th object of side A is destroyed by n-th unique shot the j-th object of side B since the time t_{k-1}, and 0 otherwise.

The probability $p_{ij}^{B}(n, t_{k-1})$ and $p_{ij}^{A}(n, t_{k-1})$ should be calculated with respect to the terrain conditions and equipments parameters.

From (27) and (28) we obtain

$$t_k^A = t_{i_k}^A(k) = \min_{i \in N_B^A(t_{k-1})} t_i^A(k) \tag{31}$$

$$t_k^B = t_{j_k}^B(k) = \min_{j \in N_A^B(t_{k-1})} t_j^B(k) \tag{32}$$

Therefore

$$t_k = \min\{t_k^A, t_k^B, \tau_k^A, \tau_k^B\} \tag{33}$$

where:

τ_k^A – the moment when for the first time since t_{k-1} side A decided to change its assignment due to other case than change of state,

τ_k^B – the moment when for the first time since t_{k-1} side B decided to change its assignment due to other case than change of state.

Introducing denotation

$$t'_k = \min\{t_k^A, t_k^B\} \tag{34}$$

the state of both sides at time t_k is equaled to

$$S(t_k) = S(t_{k-1}) \text{ when } t'_k > t_k \tag{35}$$

and for $t'_k \le t_k$

$$S(t_k) = \begin{cases} S_j^B(t_k) = S_j^B(t_{k-1}) \text{ for } j \ne j_k \\ S_{j_k}^B(t_k) = 1 \qquad \text{when } t_{j_k}^B(k) < t_{i_k}^A(k) \\ S^A(t_k) = S^A(t_{k-1}) \end{cases} \tag{36}$$

or

$$S(t_k) = \begin{cases} S_i^A(t_k) = S_i^A(t_{k-1}) \text{ for } i \ne i_k \\ S_{i_k}^A(t_k) = 1 \qquad \text{when } t_{i_k}^A(k) < t_{j_k}^B(k) \\ S^B(t_k) = S^B(t_{k-1}) \end{cases} \tag{37}$$

4. Conclusions

The presented model of attrition process control is only one of the parts of combat process. It uses parameters from other parts and delivers some features for other ones.

The obtained solutions of the assignment problems allow us to calculate the states of two participants of battle and their casualties at any time. These are two fundamental features to assess an indicator of combat success.

5. Bibliography

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Model sterowania procesem ubywania

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Istotą proponowanego modelu sterowania procesem ubywania potencjałów w walce jest rozwiązywanie sekwencji zadań przydziału celów w określonych chwilach czasowych. Chwile te są wyznaczone dla obu stron walczących. Model uwzględnia zmiany liczby środków walki oraz zamiany warunków otoczenia. Odzwierciedlone jest to poprzez zmiany parametrów problemów. Każdy ze sformułowanych problemów należy do klasy uogólnionych zadań przydziału i może nie posiadać unimodularnej macierzy współczynników ograniczeń. Wskazano przykłady opracowanych metod rozwiązywania takich zadań.

Słowa kluczowe: modelowanie matematyczne, proces ubywania, zadania przydziału.